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LETTER TO THE EDITOR

Retrieval properties of neural networks with hierarchical clustering*

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Abstract. By using the statistical mechanics formulation of Amit, Gutfreund and Sompolinsky we investigate the retrieval properties of a model for neural networks that exhibits the same organization into clusters as Dyson's hierarchical model for ferromagnetism, combined with Hebb's learning algorithm for a non-extensive number of stored patterns. We show that if the number of clusters $l \leqslant 4$ the model is able to retrieve perfectly a family of 'descendants' together with a 'pure' embedded pattern, which appear as local minima of the free energy through a series of discontinuous transitions when the temperature (noise) is reduced. The highest critical temperature is for retrieval of the embedded patterns, that occur through a second order, continuous transition and remain as the global minima of the free energy. The 'descendants' differ from the 'ancestor' in the signs of the cluster overlaps. However, when the number of partitions in clusters increases there appear 'blurred' solutions, that consist in an arbitrary mixture of 'descendants' of a given pattern and may hinder the perfect retrieval. The number $n(l)$ of mixed solutions increases exponentially with the cluster number, $n(l) \approx e^{0.45l}$, for large values of l.

A certain amount of effort has been devoted to the problem of storing and retrieving hierarchies of patterns or memories that are organized in ultrametric sequences according to their overlap, and it is a challenge to obtain the best algorithm that optimizes this process. Parga and Virasoro [1] proposed a learning algorithm to store non-orthogonal memories with the same ultrametric organization as the pure states in the SK spin glass [2], but the performance of this model has been studied only in the case of asymmetric bonds by using dynamics [3]. Almost simultaneously Feigelman and Ioffe [4] introduced a similar learning algorithm to study the relaxational dynamics of a neural network with asymmetric synaptic strength and hierarchically organized patterns. A still different mechanism was adopted in the model proposed by Gutfreund [5] to store hierarchies of biased patterns with symmetric synaptic strength. A salient feature of this model is that it preserves the compactness and simplicity of the learning rule, thus allowing for detailed analytical and numerical investigations [5]. An alternative approach was followed by Dotsenko [6], who proposed a cluster model where the stored patterns are not organized according to their overlaps in ultrametric sequences, but they are classified into hierarchies of spin clusters with given magnetizations. The intricate learning algorithm prevents us from reaching conclusive results on the storage and retrieval properties of the model [7].

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In a previous publication [8], hereafter referred to as I, we investigated a complementary problem, namely: given a neural network model that exhibits a hierarchical organization of the neurons (and not of the embedded patterns) into clusters, what would be the nature of the retrieved states? The interest in this question is not just formal, as it has been pointed out that nested clusters of neural circuits are probably present in the cerebral cortex [9].

The neural network model we discuss here has the cluster organization of Dyson's hierarchical model for ferromagnetism [10] combined with Hebb's learning algorithm, but we avoid in the following the adjective 'hierarchical' as in the neural network terminology this refers specifically to the organization of the embedded patterns, and not of the neurons. The symmetric synaptic strengths allow for a straightforward investigation of the storage and retrieval capacities of the model by using statistical mechanics methods.

It was shown in I that Dyson's prescription leads to a partition of the N sites of a neural network into $l=2^r$ clusters of N_0 neurons each, where r is a finite integer and $N=lN_0$, that are a particular realization of the hierarchical partition in cells of the sites of a spin-glass proposed by Mézard and Virasoro [2]. The neurons are represented by Ising variables $\sigma_i=\pm 1$, $i=1\ldots N$, and there are stored p patterns that consist in a given configuration of N independent random variables ξ_i^μ, $\mu=1\ldots p$, which take values ± 1 with equal probability. The hierarchical clusters are organized as follows.

(i) At the first level there is a partition in $l=2^r$ disjoint clusters of N_0 sites each and we define the cluster variables:

$$S_{1,l_1}^\mu = \sum_{N_0(l_1-1)+1}^{N_0 l_1} \xi_i^\mu \sigma_i \qquad l_1 = 1, 2, 3, \ldots, 2^r$$

together with the interaction energy:

$$\mathcal{H}(1) = -\frac{J}{N}\varepsilon^r \sum_{l_1=1}^{2^r} \sum_\mu [S_{1,l_1}^\mu]^2$$

where ε is an arbitrary positive coupling.

(ii) At the second level every two consecutive clusters are joined into a larger cluster of $N_1=2N_0$ sites and by continuing this process we have at the kth level 2^{r-k} clusters with $N_k=N_0 2^k$ sites each. The cluster variables are defined recursively by:

$$S_{k,l_k}^\mu = S_{k-1,2l_k-1}^\mu + S_{k-1,2l_k}^\mu = \sum_{N_k(l_k-1)+1}^{N_k l_k} \xi_i^\mu \sigma_i$$

$$l_k = 1, 2, \ldots, 2^{r-k} \qquad k = 1, 2, \ldots, r$$

and at every partition we associate an energy as in (i)

$$\mathcal{H}(k) = -\frac{J}{N}\varepsilon^{r-k} \sum_{l_k=1}^{2^{r-k}} \sum_\mu [S_{k,l_k}^\mu]^2.$$

The process ends when $k=r$ and all the sites are contained in a single cluster. The total energy is obtained by adding the $\mathcal{H}(k)$, then the equilibrium states are minima of the Hamiltonian:

$$\mathcal{H} = \sum_{k=1}^{r} \mathcal{H}(k)$$

that can be written in a more transparent form in terms of a learning rule:

$$\mathcal{H} = \sum_{k=0}^{r} \varepsilon^{r-k} \sum_{l_k=1}^{2^{r-k}} \sum_{i \neq i'}^{\{l_k\}} J_{i,i'}(l_k)\sigma_i\sigma_{i'}$$

where sites in a given cluster are fully connected by the synaptic efficacies:

$$J_{i,i'}(l_k) = \left[\frac{J}{N}\sum_{\mu=1}^{p} \xi_i^\mu \xi_{i'}^\mu\right]_{i,i' \in \{l_k\}}$$

and $\{l_k\}$ indicates the sites in the cluster l_k.

Dyson's hierarchical model [10] is obtained by setting $J_{i,i'}(l_k) \equiv J2^{-k}$, $N_0 = 2$, $r = \ln N/\ln 2$, and it simulates a chain of spins with decaying power law interactions $R^{-(1+\sigma)}$, with $\sigma = \ln(g)/\ln(2)$.

In order to apply the method of Amit *et al* [11] we write the Hamiltonian as

$$\mathcal{H} = -J\frac{N_0}{l}\sum_{a,b} A_{ab}(l)\sum_{\mu} S_a^\mu S_b^\mu \tag{1}$$

where the subindices $a, b = 1, 2, \ldots, l$ run over clusters and the sums:

$$S_a^\mu = \frac{1}{N_0}\sum_{i \in a} \sigma_i \xi_i^\mu \tag{2}$$

are cluster overlaps. It is understood that $N_0 \to \infty$ in the thermodynamic limit while $l = N/N_0$ remains finite. The coefficients $A_{ab}(l)$ are the elements of a $l \times l$ matrix $\mathbf{A}$ that turns out to be of Parisi's ultrametric form [12] and given by the recursion relation:

$$\mathbf{A}(l) = \begin{bmatrix} \varepsilon\mathbf{A}(l/2) + \mathbf{U}(l/2) & \mathbf{U}(l/2) \\ \mathbf{U}(l/2) & \varepsilon\mathbf{A}(l/2) + \mathbf{U}(l/2) \end{bmatrix} \tag{3}$$

where $\mathbf{U}(d)$ is a d-dimensional matrix with all elements equal to unity and $\mathbf{A}(1) = 1$. Hopfield's model is recovered for $\varepsilon = 0$, $A_{ab}(l) \equiv 1$.

For a finite number p of embedded patterns we follow the standard procedure first proposed by Amit *et al* [11] and we obtain from the Hamiltonian of (1) the free energy density:

$$\frac{\beta F}{N_0} = \frac{1}{2}\frac{\beta J}{l}\sum_{a,b} A_{ab}\boldsymbol{m}_a \cdot \boldsymbol{m}_b - \sum_a \left\langle \log\cosh\left(\frac{\beta J}{l}\sum_b A_{ab}\boldsymbol{\xi}_i \cdot \boldsymbol{m}_b\right)\right\rangle_{\{a\}} \tag{4}$$

where the 'temperature' $T = \beta^{-1}$ is a measure of synaptic noise and $\boldsymbol{m}_a = \{m_a^\mu\}$, $\boldsymbol{\xi}_i = \{\xi_i^\mu\}$, $\mu = 1, \ldots, p$, are p-component vectors. In the following we avoid writing the block number 'l' in the matrix elements of $\mathbf{A}$ unless explicitly needed for clarity. The quantities m_a^μ are the thermal averages of the cluster overlaps in (2):

$$m_a^\mu = \frac{1}{N_0}\sum_{i \in a} \xi_i^\mu \langle\sigma_i\rangle_{\mathrm{TA}} = \langle \xi_i^\mu \langle\sigma_i\rangle_{\mathrm{TA}}\rangle_{\{a\}} \tag{5}$$

where we use the notation $\langle \ldots \rangle_{\{a\}}$ to indicate the configurational average over the random variable ξ_i^μ when i belongs to cluster a. The last equality in (5) follows from self-averaging as the cluster size N_0 grows to infinity in the thermodynamic limit.

The overlaps are determined from the saddle-point equations:

$$\boldsymbol{m}_a = \left\langle \boldsymbol{\xi}_i \tanh\left[\frac{\beta J}{l}\sum_b A_{ab}\boldsymbol{\xi}_i \cdot \boldsymbol{m}_b\right]\right\rangle_{\{a\}} \tag{6}$$

and the stability of these solutions is determined by the eigenvalues of the pl-dimensional stability matrix with elements:

$$M_{\mu a,\nu b}=\delta_{\mu\nu}A_{ab}-\frac{\beta J}{l}\sum_c A_{ac}\left\{\delta_{\mu\nu}-\left\langle \xi^\mu\xi^\nu \tanh^2\left[\frac{\beta J}{l}\sum_d A_{cd}\boldsymbol{\xi}\cdot\boldsymbol{m}_d\right]\right\rangle_{\{c\}}\right\}A_{cb}. \tag{7}$$

Retrieval solutions. The solutions of (6) with perfect retrieval along one of the patterns, say $\mu=1$, are of the form $m_a^\mu=\delta^{\mu 1}m_a$ where m_a satisfies:

$$m_a=\tanh\left[\frac{\beta J}{l}\sum_b A_{ab}m_b\right]. \tag{8}$$

The simplest solution to (8) occurs when m_a aligns along one of the eigenvectors $\bar{v}^\gamma$ of $\mathbf{A}$ with eigenvalue λ_γ:

$$m_a^\gamma=m_\gamma v_a^\gamma \tag{9}$$

where we impose the normalization $v_a^\gamma=\pm 1$; then it follows that

$$m_\gamma=\tanh\left[\beta J\frac{\lambda_\gamma}{l}m_\gamma\right]. \tag{10}$$

The eigenvectors and eigenvalues of $\mathbf{A}(l)$ were discussed in detail in I and have the following properties.

(a) They can be obtained through the recursion relations:

$$\begin{aligned}&\bar{v}^{2\eta-1}(l)=\begin{bmatrix}\bar{v}^\eta(l/2)\\ \bar{v}^\eta(l/2)\end{bmatrix}\qquad \bar{v}^{2\eta}(l)=\begin{bmatrix}\bar{v}^\eta(l/2)\\ -\bar{v}^\eta(l/2)\end{bmatrix}\\ &\bar{v}^1=1\qquad \eta=1,2,\ldots,\frac{l}{2}\end{aligned} \tag{11}$$

with the corresponding eigenvalues:

$$\begin{aligned}&\lambda_1(l)=l+\varepsilon\lambda_1\left(\frac{l}{2}\right)\\ &\lambda_2(l)=\varepsilon\lambda_1\left(\frac{l}{2}\right)\\ &\lambda_{2\eta-1}(l)=\lambda_{2\eta}(l)=\varepsilon\lambda_\eta\left(\frac{l}{2}\right)\qquad \eta\geqslant 2.\end{aligned} \tag{12}$$

(b) Except for $\bar{v}^1(l)$ and $\bar{v}^2(l)$, all the other eigenvectors fall into degenerate groups with decreasing eigenvalues:

$$\lambda_1>\lambda_2>\lambda_3=\lambda_4>\ldots>\lambda_{l/2+1}=\lambda_{l/2+2}=\ldots=\lambda_l. \tag{13}$$

(c) Except for $\bar{v}^1(l)$, that has all components equal to unity, all the other $\bar{v}^\gamma(l)$, $\gamma\geqslant 2$, are eigenvectors of $\mathbf{U}(l)$ in (3) with vanishing eigenvalue.

We represent in figure 1 the eigenvectors for the special case $l=8$.

To take into account the degeneracy in (13), it is convenient to introduce in (10) the notation:

$$\begin{aligned}&\beta_1=\frac{l}{J\lambda_1}\\ &\beta_s=\frac{l}{J\lambda_s}\qquad 2^{s-2}\leqslant\gamma<2^{s-1}\qquad s=2,\ldots,r+1.\end{aligned} \tag{14}$$

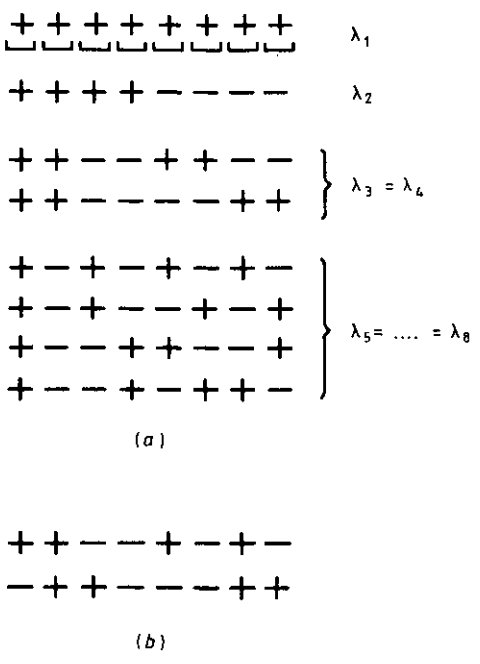

Figure 1. (*a*) We display on each row the components $v_a^\gamma = \pm 1$ of the eigenvectors of **A** in (13) for $l = 8$, together with the corresponding degenerate eigenvalue. The top eigenvector with λ_1 has a perfect alignment with the embedded pattern, while the 'descendants' differ in the orientation of the cluster magnetization. (*b*) Examples of a 'mixed' retrieval solution, where half the number of clusters aligns with a vector of the second group (λ_3) and the other half with a vector of the third group (λ_5).

The non-trivial solutions to (10), together with the positiveness condition on the eigenvalues of (7) show that as the temperature decreases the system undergoes the following series of ordering transitions [8]:

(i) $\beta < \beta_1$, $m_\gamma = 0$ for all γ (paramagnetic phase)

(ii) $\beta_1 \leq \beta < \beta_s$, $s = 2, 3, \ldots, r$.

The system of equations (10) accepts a solution

$$m_1 \neq 0 \qquad m_\gamma = 0 \qquad \text{for } \gamma \geq 2.$$

This is a continuous transition with m_1 growing continuously from zero as $m_1 \approx (\beta/\beta_1 - 1)^{1/2}$.

(iii) $\beta \geq \beta_s$, $s = 2, \ldots, r$.

We show in I that (10) accepts other non-trivial solutions but the transition cannot be continuous because one has to keep positive the lowest eigenvalue of the stability matrix in (7). Each group of 'descendants' eigenvectors with $2^{s-2}+1 \leq \gamma \leq 2^{s-1}$ orders at $\beta = \beta_s^* > \beta_s$ with a discontinuity $m_\gamma = m_s^*$ in the order parameter given by the joint solution of the equations:

$$\begin{aligned} m_s^* &= \tanh\left(\frac{\beta_s^*}{\beta_s} m_s^*\right) \\ \frac{\beta_s^*}{\beta_1}(1 - m_s^{*2}) &= 1. \end{aligned} \tag{15}$$

At any finite temperature these solutions are local minima of the free energy, as we obtain from (4), (9) and (10) that at the saddle point the free energy density can be written:

$$\frac{F_s(\beta)}{N_0} = -\frac{1}{\beta_s} g\left(\frac{\beta}{\beta_s}\right) \tag{16}$$

where $g(x)$ is a monotonous increasing function, then we will have $F_1 < F_2 < \ldots < F_r$, with the solution $m_1 \neq 0$ being the global minimum. At zero temperature we obtain from (8) and (4):

$$\frac{E}{N_0} = -\frac{J}{2l} \sum_a \left| \sum_b A_{ab} m_b \right| \tag{17}$$

where the sum on the right-hand side satisfies the inequality:

$$\frac{1}{l} \sum_a \left| \sum_b A_{ab} m_b \right| \leqslant \left[\frac{1}{l} \sum_a \left(\sum_b A_{ab} m_b \right)^2 \right]^{1/2}. \tag{18}$$

The equals sign in (18) applies only to the retrieval solution m_a^γ in (9), that gives:

$$\left| \sum_b A_{ab} m_b^\gamma \right| = |\lambda_\gamma v_a^\gamma| = \lambda_\gamma \tag{19}$$

from where it follows, together with (17), that $E_1 < E_2 < \ldots < E_r$.

We conclude that the 'true retrieval' solutions with $\gamma = 1$ and $v_a^1 = 1$, that align perfectly with the embedded patterns give at all values of $\beta \geqslant$ the global minima of the free energy. However, the other solutions in (9) with $\gamma \geqslant 2$ that form a 'family of descendants' that differ in the relative signs of the cluster overlaps, as shown in figure 1(*a*), will be local minima and possible attractors in a relaxational process.

Non-aligned retrieval solutions. Equation (8) also accepts the mixed solution:

$$m_c^{\gamma\gamma'} = \begin{cases} m_\gamma v_c^\gamma(l/2) & 1 \leqslant c < l/2 \\ m_{\gamma'} v_c^{\gamma'}(l/2) & l/2 \leqslant c < l \end{cases} \tag{20}$$

where the vectors $\bar{v}^\gamma(l/2)$ are eigenvectors of $\mathbf{U}(l/2)$ with vanishing eigenvalue and from (3):

$$\begin{aligned} m_\gamma &= \tanh\left[\beta J \frac{\varepsilon \lambda_\gamma}{l} \left(\frac{l}{2}\right) m_\gamma \right] \\ m_{\gamma'} &= \tanh\left[\beta J \frac{\varepsilon \lambda_{\gamma'}}{l} \left(\frac{l}{2}\right) m_{\gamma'} \right]. \end{aligned} \tag{21}$$

We fix arbitrarily $\lambda_\gamma > \lambda_{\gamma'}$, then $m_\gamma > m_{\gamma'}$. We show in figure 1(*b*) two examples of 'mixed' solutions for $l = 8$.

For these solutions the stability matrix in (7) becomes:

$$M_{1a,1b}^{(\gamma\gamma')} = A_{ab} - \frac{\beta J}{l} \sum_c A_{ac}[1 - (m_c^{\gamma\gamma'})^2] A_{cb} \tag{22}$$

and can be brought into block diagonal form by means of a unitary transformation:

$$\sum_{a,b} v_a^{\eta'} M_{1a,1b}^{(\gamma\gamma')} v_b^\eta = \lambda_{\eta'} \Lambda_{\eta'\eta} \tag{23}$$

where the matrix Λ:

$$\Lambda = \begin{bmatrix} \omega_1 & \sigma_{12} & & 0 & & & & \\ \sigma_{21} & \omega_2 & & & & 0 & & \\ & & \omega_3 & \sigma_3 & & & & \\ & 0 & \sigma_3 & \omega_3 & & & & \\ & & & & \omega_4 & \sigma_4 & & 0 \\ & & 0 & & \sigma_4 & \omega_4 & & \\ & & & & & & \omega_4 & \sigma_4 \\ & & & & 0 & & \sigma_4 & \omega_4 \end{bmatrix} \tag{24}$$

has the non-vanishing matrix elements:

$$\begin{aligned} &\Lambda_{\eta\eta} = \omega_s = 1 - \frac{\beta}{\beta_s}\left[1 - \frac{m_\gamma^2 + m_{\gamma'}^2}{2}\right] \\ &\Lambda_{\eta'\eta} = \frac{\beta}{\beta_s}\left(\frac{m_\gamma^2 - m_{\gamma'}^2}{2}\right) = \sigma_{\eta'\eta} \\ &\eta' = 2\delta - 1 \quad \eta = 2\delta \qquad \text{or} \qquad \eta' = 2\delta \quad \eta = 2\delta - 1 \\ &2^{s-2} + 1 \leqslant \eta \leqslant 2^{s-1} \qquad \text{for } s = 2, 3, \ldots, r \\ &\eta = 1 \qquad \text{for } s = 1. \end{aligned} \tag{25}$$

According to (23), as $\lambda_\eta \geqslant 0$, it is a sufficient condition for the stability of the solutions in (20) that they should keep positive the eigenvalues of Λ:

$$\begin{aligned} &\Lambda_1^{\mp} = \tfrac{1}{2}(\omega_1 + \omega_2) \mp \tfrac{1}{2}\{(\omega_1 - \omega_2)^2 + 4\sigma_{12}\sigma_{21}\}^{1/2} \\ &\Lambda_s^- = 1 - \frac{\beta}{\beta_3}(1 - m_{\gamma'}^2) \\ &\Lambda_s^+ = 1 - \frac{\beta}{\beta_s}(1 - m_\gamma^2) \end{aligned} \tag{26}$$

with $\Lambda_1^- < \Lambda_1^+ < \Lambda_3^- < \Lambda_3^+ < \ldots$.

To impose the condition of a non-trivial solution to both equations (21) together with the positiveness of the lowest eigenvalue Λ_1^- in (26), provides us with three equations for m_γ^*, $m_{\gamma'}^*$ and the critical $\beta_{\gamma\gamma'}^*$ for the discontinuous transition.

For these solutions the inequality sign in (18) is valid, which indicates imperfect alignment. From (20), (4) and the recursion relation for the eigenvalues in (12), there follows the expression for the free energy:

$$\frac{1}{N} F_{\gamma\gamma'}(\beta) = \frac{1}{2N}(F_\gamma(\beta) + F_{\gamma'}(\beta)). \tag{27}$$

Numerical results for T_{34}^* as a function of the parameter ε in (12) are shown by the broken curve in figure 2, for $l = 8$, $s = 3$, $s' = 4$, and γ, γ' as in (14). The full curves represent the critical temperatures T_3^* (upper) and T_4^* (lower) for the retrieval solutions in (15).

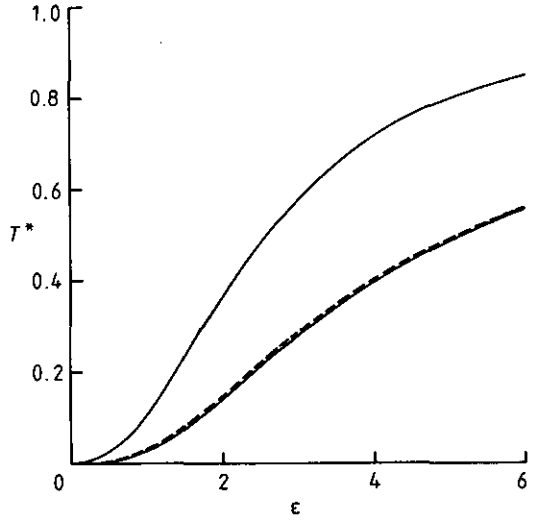

Figure 2. Numerical results for the critical T_{34}^{*} (broken line) of the 'mixed' solution in figure 1(*b*) as a function of the interaction strength ε in (3). The full curves represent the critical temperatures T_3^{*} (upper) and T_4^{*} (lower) for the retrieval of the 'descendants' in (15). We fixed the scale $l/J\lambda_1=\beta_1=1$.

We observe that the critical T_{34}^{*} is slightly higher but almost indistinguishable from T_4^{*}, then the 'mixed' solution would order approximately at the same temperature as the lowest 'pure' solution with a lower free energy, as we get from (27): $F_3<F_{34}<F_4$. This seems to be an indication that, if the number of 'imperfect' solutions is sufficiently large, the retrieval properties of the model may be severely affected because the 'mixed' minima will be more attractive than one of the 'pure' components.

To estimate the number of 'mixed' solutions we observe that the choice in (20) does not exploit all the possibilities, and we may also have as solutions of (8):

$$m_c^{\gamma\eta\mu\nu}=\begin{cases} m_\gamma v_c^\gamma(l/4) & 1\leqslant c<l/4 \\ m_\eta v_c^\eta(l/4) & l/4\leqslant c<l/2 \\ m_\mu v_c^\mu(l/4) & l/2\leqslant c<3l/4 \\ m_\nu v_c^\nu(l/4) & 3l/4\leqslant c<l \end{cases} \tag{28}$$

where the $(l/4)$-dimensional vectors $\bar{v}^\gamma(l/4)$ are eigenvectors of $\mathbf{U}(l/4)$ with vanishing eigenvalue. This partition process can be continued and in general after s steps we will have a solution in terms of 2^s eigenvectors $\bar{v}^\gamma(l/2^s)$ of $\mathbf{U}(l/2^s)$ with vanishing eigenvalue, for $s=1,2,\ldots,r$, $l=2^r$.

We call $N(l)$ the number of vectors $\bar{w}(l)$ that satisfy:

$$\mathbf{U}(l)\bar{w}(l)=0 \tag{29}$$

and that are not the eigenvectors in (11). The vectors $\bar{w}(l)$ can be obtained through the recursion relations

$$\bar{w}(l)=\begin{pmatrix} \bar{w}'(l/2) \\ \pm\bar{w}''(l/2) \end{pmatrix} \tag{30}$$

where $\bar{w}'(l/2)$ satisfies the same condition as in (29) with l replaced by $l/2$. Then with

each pair of $(l/2)$-dimensional vectors we can build two l-dimensional ones, and to this we must add

$$\bar{w}_2(l)=\begin{pmatrix} 1 \\ \vdots \\ 1 \\ -1 \\ \vdots \\ -1 \end{pmatrix} \begin{matrix} \} \frac{l}{2} \\ \\ \} \frac{l}{2} \end{matrix} \tag{31}$$

that satisfies (29) but is not in the form of (30). Among all the $\bar{w}(l)$ built in this way we will obtain also the $(l-1)$ eigenvectors that satisfy (29); then we get:

$$N(l)+(l-1)=2\left[N\left(\frac{l}{2}\right)+\left(\frac{l}{2}-1\right)\right]^2+1 \tag{32}$$

with $N(1)=0$. The recursion relation in (32) gives $N(2)=N(4)=0$, $N(8)=12$ and for large values of l has the asymptotic behaviour:

$$N(l)\approx e^{0.45l}-l \tag{33}$$

thus growing exponentially with the number of clusters. We conclude that the present model with hierarchical clustering has excellent retrieval properties for one embedded pattern and a 'family of descendants' that vary in the relative sign of the cluster overlaps, if the number of clusters is kept relatively small ($l<8$).

When the number of partitions increases, there appear an exponentially large number of 'blurred', unwanted solutions that are not aligned with the embedded patterns or its 'pure' descendants. These mixed solutions would have a lower free energy than those corresponding to full alignment of the pure component with lowest eigenvalue, thus acting as more effective attractors in a relaxational process and preventing perfect retrieval.

To conclude, we present in this work a neural network model where some notion of 'nearness' is defined, in such a way that we can speak about the hierarchical organization of the neurons in clusters. Explicit calculations show that for each embedded pattern the system retrieves not only the pattern itself but also a family of descendants. Although our model is reminiscent of cluster models studied in the past [6, 9], it is far more tractable and allows for a detailed investigation of its storing and retrieving capacities, hence the results we present here go beyond the information presented in previous papers.

The saturation properties of the model will be analysed in a forthcoming publication.

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